Space–time singularities and the axion in the Poincaré coset models ISO(2,1)/H

Roberto Casadio* and Benjamin Harms†

Department of Physics and Astronomy, The University of Alabama
Box 870324, Tuscaloosa, AL 35487-0324

February 1, 2008

Abstract

By promoting an invariant subgroup H of $ISO(2,1)$ to a gauge symmetry of a WZWN action, we obtain the description of a bosonic string moving either in a curved 4-dimensional space–time with an axion field and curvature singularities or in 3-dimensional Minkowski space–time.

Pacs: 04.20.DW, 11.25.-w, 11.30.Cp

Keywords:

axion field

bosonic strings

coset construction

curved space–times

Poincaré invariance

signature changes

space–time singularities

spinning strings

UAHEP 965

*Permanent Institution: I.N.F.N., Sezione di Bologna, Italy, e–mail: casadio@bologna.infn.it

†e–mail: bharms@ua1vm.ua.edu

1 Introduction

In recent years it has emerged that several string actions naturally describe curved space–times with singularities. This was first realized when Witten discovered that a gauged WZWN action for $SL(2,\mathbb{R})/U(1)$ contains a black hole [1]. In a subsequent paper [2], this model was extended and found to be conjugated to a black string, and more general coset models G/H, with G a simple non–compact group, have been analyzed [3].

In this letter we apply the coset construction of Refs. [1, 2, 3] to a WZWN action in the Poincaré group $ISO(2,1)$ that describes a closed bosonized spinning string in 2+1-dimensional Minkowski space–time [4] and we argue that it leads to an effective theory with $6-\dim(H)$ degrees of freedom.

We also show, in the framework of a particular parameterization, that one further degree of freedom can be eliminated, giving the description of a string without spin moving in a 4-dimensional curved space–time with an axion field and curvature singularities or in 3-dimensional Minkowski space–time.

2 The gauged WZWN action

The elements of the Poincaré group $ISO(2,1)$ can be written using the notation $g=(\Lambda,v)$, where $\Lambda\in SO(2,1)$ and $v\in\mathbb{R}^3$. Given the map $g:\ M=D^2\times R\mapsto ISO(2,1)$ from the 2-dimensional disc$\times$time to $ISO(2,1)$, we consider the WZWN action

$$S=\frac{1}{2\lambda^2}\int_{\partial M}d^2\sigma\,\langle\,g^{-1}\,dg,g^{-1}\,dg\,\rangle+\frac{n}{12\,\pi}\int_M\langle\,g^{-1}\,dg,(g^{-1}\,dg)^2\,\rangle\ , \tag{1}$$

where ∂M is the boundary of the manifold M, that is the string world–sheet parameterized by the light–cone coordinates σ^+,σ^-, and the brackets $\langle\,,\,\rangle$ denote the non–degenerate bilinear invariant of $ISO(2,1)$. If $-1/\lambda^2=n/4\,\pi\equiv\kappa/2$, the action can be written entirely on the boundary and it describes a closed bosonized spinning string moving in 2+1 Minkowski space–time with coordinates v^k, $k=0,1,2$ [4],

$$S=-\frac{\kappa}{4}\int_{\partial M}d^2\sigma\,\epsilon^{ijk}\left(\partial_+\Lambda\,\Lambda^{-1}\right)_{ij}\,\partial_-v_k\ , \tag{2}$$

where ϵ^{ijk} is the 3-dimensional Levi–Civita symbol.

The action in Eq. (2) is invariant under $g\mapsto h_{_L}(\sigma^+)\,g\,h_{_R}^{-1}(\sigma^-)$, where $h_L,\,h_R\in ISO(2,1)$, and also under the left and right action of the group of diffeomorphisms of the world–sheet [4]. However, it is not invariant under the local action of any subgroup H of $ISO(2,1)$ given by $g\mapsto$

$h_L\, g\, h_R^{-1} = \left(\theta_L\, \Lambda\, \theta_R^{-1}, -\theta_L\, \Lambda\, \theta_R^{-1}\, y_R + \theta_L\, v + y_L\right)$, where $h_{L/R} = h_{L/R}(\sigma^-, \sigma^+) = (\theta_{L/R}, y_{L/R}) \in H$, due to the dependence of h_L on σ^- and of h_R on σ^+. To promote H to a gauge symmetry of the action we introduce gauge fields $A_\pm = (\omega_\pm, \xi_\pm) \in iso(2,1)$, the Lie algebra of the Poincaré group, and covariant derivatives $D_\pm = \partial_\pm + A_\pm$.

We also demand H to be invariant, such that $\delta g = h_L\, g\, h_R^{-1} \in H$. The only possible choices for $ISO(2,1)$ are subgroups of the translation group $\mathbb{R}^3$, that is $h_{L/R} = (0, y^n_{L/R})$, where n runs in a subset of $\{0,1,2\}$, for which $\delta g = (0, y_L) = h_L\, g$. In this case $\omega_\pm \equiv \xi_+ \equiv 0$, and $\xi_-^k \equiv 0$ iff the translation in the k direction is not included in H. The gauged action then reads

$$S_g = -\frac{\kappa}{4} \int d^2\sigma\, \epsilon^{ijk} \left(\partial_+ \Lambda\, \Lambda^{-1}\right)_{ij} (\partial_- v + \xi_-)_k \ . \tag{3}$$

For the ungaged action S in Eq. (2) the equations of motion $\delta_v S = 0$, which follow from the variation $v \to v + \delta v$, with δv an infinitesimal 2+1 vector, lead to the conservation of the momentum currents $P_+^k \equiv \epsilon^{ijk} (\partial_+ \Lambda\, \Lambda^{-1})_{ij}$, $k = 0,1,2$ [4]. In the gauged case this variation must be supplemented by the condition that the gauge field varies under an infinitesimal $ISO(2,1)$ transformation according to

$$\xi_-^n \to \xi_-^n - \partial_-(\delta v^n) \ , \tag{4}$$

and from $\delta_v S_g = 0$ one obtains

$$\partial_- P_+^{k\neq n} = 0 \ , \tag{5}$$

so that only the $P_+^{k\neq n}$ currents are still conserved.

Similarly, from the variation $\Lambda \to \Lambda + \delta\Lambda$, $\delta\Lambda = \Lambda\, \epsilon$ and $\delta v = \epsilon\, v$, with $\epsilon_{ij} = -\epsilon_{ji}$ an infinitesimal $so(2,1)$ matrix, the equations $\delta_\epsilon S = 0$ lead to the conservation of the three angular momentum currents $J_-^k \equiv (\Lambda\, \partial_- v)^k$, $k = 0,1,2$, which can be shown to include a contribution of intrinsic (non orbital) spin [4]. In the gauged case, by making use of Eq. (4) and Eq. (5), one obtains

$$\partial_+ J_-^k = -\partial_+ (\Lambda\, \xi_-)^k \ , \tag{6}$$

so that the currents J_-^k couple to the gauge field.

We are free to choose $\dim(H)$ gauge conditions to be satisfied by the elements of $ISO(2,1)/H$. It appears natural to impose

$$\xi_-^n = -\partial_- v^n \ , \tag{7}$$

so that the previous equations of motion become the same as $\delta_v S_{eff} = \delta_\epsilon S_{eff} = 0$ obtained by varying the effective action

$$S_{eff} = -\frac{\kappa}{2} \int d^2\sigma \sum_{k\neq n} P_+^k\, \partial_- v_k \ , \tag{8}$$

where the sum runs only over the indices corresponding to the translations not included in H.

We observe that, although the number of degrees of freedom in the effective action is $\dim(ISO(2,1)) - \dim(H) = 6 - \dim(H)$ if $\dim(H) < 3$, gauging the whole 3-dimensional translation subgroup makes the effective theory empty and the present reduction scheme fails. Thus we shall attempt at gauging 1-and 2-dimensional translations only.

2.1 Gauging a 1-dimensional translation

In order to obtain an explicit form for the momentum currents we write an $SO(2,1)$ matrix $\Lambda^i{}_j$ as a product of two rotations (of angles α and γ) and a boost (β) [5],

$$\begin{aligned}
\Lambda &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cosh\beta & 0 & \sinh\beta \\ 0 & 1 & 0 \\ \sinh\beta & 0 & \cosh\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \\
&= \begin{bmatrix} \cosh\beta & \sinh\beta\,\sin\gamma & \sinh\beta\,\cos\gamma \\ -\sin\alpha\,\sinh\beta & \cos\alpha\,\cos\gamma - \sin\alpha\,\cosh\beta\,\sin\gamma & -\cos\alpha\,\sinh\beta - \sin\alpha\,\cosh\beta\,\sin\gamma \\ \cos\alpha\,\sinh\beta & \sin\alpha\,\cos\gamma + \cos\alpha\,\cosh\beta\,\sin\gamma & -\sin\alpha\,\sin\gamma + \cos\alpha\,\cosh\beta\,\cos\gamma \end{bmatrix}
\end{aligned} \tag{9}$$

and we obtain

$$\begin{aligned}
P^0_+ &= -\partial_+\alpha - \cosh\beta\,\partial_+\gamma \\
P^1_+ &= -\cos\alpha\,\partial_+\beta - \sin\alpha\,\sinh\beta\,\partial_+\gamma \\
P^2_+ &= -\sin\alpha\,\partial_+\beta + \cos\alpha\,\sinh\beta\,\partial_+\gamma\ .
\end{aligned} \tag{10}$$

Then we choose $H = \{(0, y^0)\}$ and, since no derivative of α occurs in P^1_+ and P^2_+, we also rotate the variables v^1 and v^2 by an angle $-\alpha$,

$$\begin{bmatrix} \partial_-\tilde{v}^1 \\ \partial_-\tilde{v}^2 \end{bmatrix} \equiv \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \partial_- v^1 \\ \partial_- v^2 \end{bmatrix}\ . \tag{11}$$

This can be considered as an internal symmetry of the effective theory which is used to further simplify the effective action in Eq. (8) with $n = 0$ to the form

$$S_{eff} = -\frac{\kappa}{2}\int d^2\sigma\,\left[-\partial_+\beta\,\partial_-\tilde{v}^1 + \sinh\beta\,\partial_+\gamma\,\partial_-\tilde{v}^2\right]\ . \tag{12}$$

Now we can introduce new string coordinates x^i, $i = 1, \ldots, 4$, defined by

$$\begin{cases} 4\,x^1 = \beta + \gamma + v^1 + v^2 \\ 4\,x^2 = \beta - \gamma + v^1 + v^2 \\ 4\,x^3 = \beta + \gamma - v^1 + v^2 \\ 4\,x^4 = \beta + \gamma + v^1 - v^2\ , \end{cases} \tag{13}$$

and the action in Eq. (12) finally becomes

$$S_{eff} = -\frac{\kappa}{4}\int d^2\sigma\,\left[G_{ij}\,\partial_+x^i\,\partial_-x^j + B_{ij}\,(\partial_+x^i\,\partial_-x^j - \partial_-x^i\,\partial_+x^j)\right]\ , \tag{14}$$

where $(f(\beta) \equiv \sinh\beta)$

$$G_{ij}(\beta) = \begin{bmatrix} 2\,(f-1) & -1 & 0 & -1 \\ -1 & -2\,(f+1) & 0 & (f-1) \\ 0 & 0 & 2\,(f+1) & 0 \\ -1 & (f-1) & 0 & -2\,(f+1) \end{bmatrix} . \tag{15}$$

and it can be regarded as the metric tensor of our 4-dimensional space–time. The matrix $B_{ij} = B_{ij}(\beta)$ is antisymmetric and defined by

$$\begin{array}{ll} B_{12} = f & B_{23} = 1 - f \\ B_{13} = 1 & B_{24} = 0 \\ B_{14} = -f & B_{34} = -(f+1) \ . \end{array} \tag{16}$$

It represents an axion field whose field strength, $H_{ijk} \equiv \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}$, has $H_{124} = 2\cosh\beta$ as the only non–zero component.

One can also show that the Ricci tensor R_{ij} is not zero and the curvature scalar,

$$R(\beta) = \frac{27\,f^7 + 186\,f^6 + 49\,f^5 - 179\,f^4 + 87\,f^3 + 16\,f^2 - 17\,f - 3}{(3\,f+1)^2\,(f+1)^2\,(f^2+2\,f-2)^2} \ , \tag{17}$$

has singularities in $f \equiv \sinh\beta = -1, -1/3, -1 \pm \sqrt{3}$ (see Fig. 1 for a plot of $R(\beta)$). Thus $\mathbf{G}$ is not a vacuum solution of the gravitational equations, and the action in Eq. (14) is no longer conformally invariant. A dilaton field Φ must be introduced according to the general 1-loop expression $R_{ij} = \nabla_i \nabla_j \Phi$, plus the central term $(d-26)/3 = 22/3$ due to the dimension $d = 4$ of space–time [1, 6].

The signature of the metric is $3+1$ in a neighborhood of $\beta = 0$, but it changes in coincidence with the curvature singularities, as can be inferred by noting that the determinant,

$$G \equiv \det(\mathbf{G}) = 4\,(3\,f+1)\,(f+1)\,(f^2+2\,f-2) \ , \tag{18}$$

is proportional to the square root of the denominator of R (see Fig. 2 for a plot of $G(\beta)$).

2.2 Gauging 2-dimensional translations

We choose to gauge $H = \{(0, y^1), (0, y^2)\}$. Since no derivative of β occurs in P^0_+, by defining $\partial_- w \equiv \cosh\beta\,\partial_-\gamma$, we can eliminate it. Introducing $3\,x^1 \equiv \alpha + w + v^0$, $3\,x^2 \equiv \alpha - w + v^0$ and $3\,x^3 \equiv \alpha + w + v^0$ the effective action in Eq. (8) with $k = 0$ can be written in the same form as Eq. (14), but now G_{ij} is a constant symmetric matrix with signature 2+1 and B_{ij} is a constant antisymmetric matrix.

Again, if we interprete x^1, x^2, x^3 as the coordinates of a string, we conclude that the target space is the same 2+1 Minkowski space–time of the ungaged action in Eq. (2), but the string has lost its intrinsic spin and the axion field is a pure gauge, $H_{ijk} \equiv 0$.

3 Conclusions

One of the main aspects of the model we have studied is that its effective action naturally contains an axion field together with a bosonic string in (possibly) curved backgrounds.

In the 3-dimensional case that we have studied, the outcome of the coset construction seems to be quite trivial, leading solely to the initial flat space–time with the string now deprived of its intrinsic spin. However the 4-dimensional case shows regions of different signatures (including Minkowskian regions) separated by curvature singularities.

Further, by gauging different translations one can obtain (5- and 4-dimensional) solutions other than the ones we have shown here. We are at present trying to perform a complete analysis of the model, including the use of different parameterizations of the Lorentz group.

Acknowledgements

This work was supported in part by the Department of Energy under contract n. DE-FG02-96ER40967. We would like to thank A. Stern for many useful discussions.

References

[1] E. Witten, Phys. Rev D 44 (1991) 314.

[2] J. Horne and G. Horowitz, Nucl. Phys. **B368** (1992) 444.

[3] I. Bars and K. Sfetsos, Phys. Rev. D 46 (1991) 4495.

[4] A. Stern in *Superstrings and Particle theory*, ed. L. Clavelli and B. Harms (1990) World Scientific, Singapore; P. Salomonson, B. Skagerstam and A. Stern, Nucl. Phys. **B347** (1990) 769.

[5] A. Hanson, T. Regge and C. Teitelboim, *Constrained hamiltonian systems* (1976) Accademia Nazionale dei Lincei, Roma.

[6] M. Green, J. Schwarz and E. Witten, *Superstring theory* (1987) Cambridge Univ. Press, Cambridge.

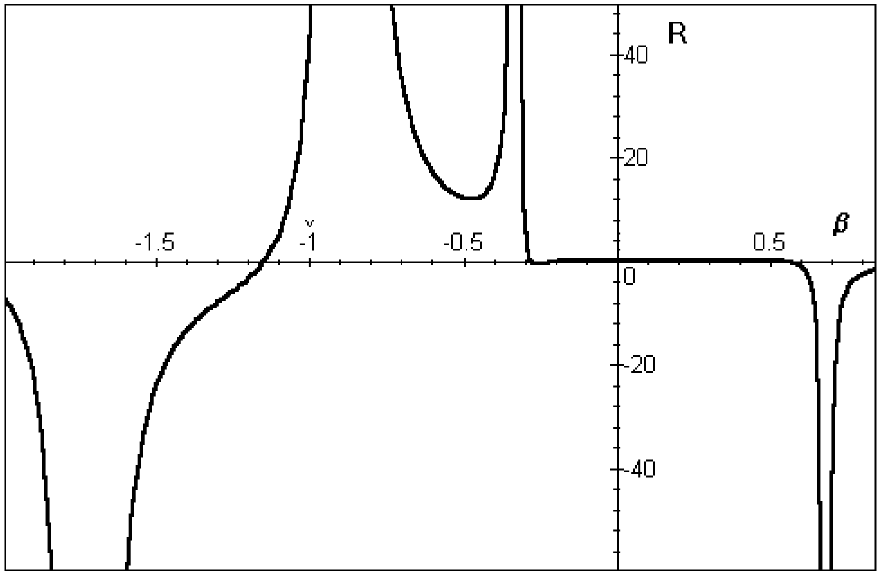

Fig. 1

Plot of the scalar curvature R given in Eq. (17). It clearly shows four singularities along the β–axis.

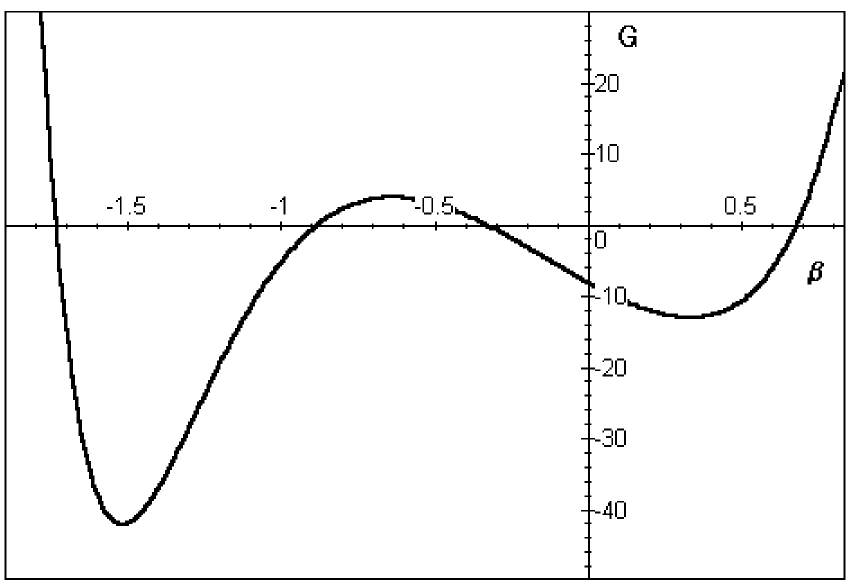

Fig. 2

Plot of the determinant G given in Eq. (18). The zeros along the β-axis coincide with the singularities of R in Fig.1. The regions in which G is negative have signature 3+1.